Quantum State & Process Tomography with a Dispersive Readout

August 8th, 2012

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Part 1: Quantum State Tomography (QST)

Section 1a.

Density Matrix Reconstruction

Density Operators

$$\boxed{\rho \equiv \sum_j P_j |\psi_j\rangle\langle\psi_j|}$$

(Hermitian)
$\text{Trace}\{\rho\} = 1$

Common basis

$$\rho_{uv} = \langle\phi_u|\rho|\phi_v\rangle = \overline{C_u C_v^*} \quad \text{for} \quad |\psi_j\rangle = \sum_u C_u^{(j)} |\phi_u\rangle$$

$$\overline{\langle\hat{\Theta}\rangle} = \sum_j P_j \langle\psi_j|\hat{\Theta}|\psi_j\rangle \quad \text{(ensemble average)}$$
$$= \text{Tr}\{\rho\hat{\Theta}\}$$

$$\rho_m = \frac{M_m \rho M_m^\dagger}{\text{Tr}\{M_m^\dagger M_m \rho\}}$$

a unitary evolution

The set of measurement operators must satisfy the completeness relation: $\sum_m M_m^\dagger M_m = I$

The state space of a composite physical system is the tensor product of the state spaces of the component physical systems:

$$\rho_{TOT} = \rho_1 \otimes \rho_2 \otimes \cdots \rho_n$$

As can be shown (ref. Nielsen and Chuang page 105)

$$\rho = \frac{I + \vec{r} \cdot \vec{\sigma}}{2}$$

$\vec{r}$ = Bloch Vector

$\vec{\sigma} = \underline{e_x}\sigma_x + \underline{e_y}\sigma_y + \underline{e_z}\sigma_z$

Expanding…

$$\rho = \frac{1}{2}\left[1\hat{I} + r_x\widehat{\sigma_x} + r_y\widehat{\sigma_y} + r_z\widehat{\sigma_z}\right]$$

$$\rho = \frac{1}{2}\left[Tr\{\hat{I}\rho\}\hat{I} + Tr\{\widehat{\sigma_x}\rho\}\widehat{\sigma_x} + Tr\{\widehat{\sigma_y}\rho\}\widehat{\sigma_y} + Tr\{\widehat{\sigma_z}\rho\}\widehat{\sigma_z}\right]$$

$$\boxed{\rho = \frac{1}{2}\left[\left\langle\bar{I}\right\rangle\hat{I} + \langle\overline{\sigma_x}\rangle\sigma_x + \langle\overline{\sigma_y}\rangle\sigma_y + \langle\overline{\sigma_z}\rangle\sigma_z\right]}$$

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Thus for a two qubit system:

$$\rho_{TOT} = \rho_1 \otimes \rho_2 = \left[\sum_R \frac{\overline{\langle R \rangle} R}{2}\right] \otimes \left[\sum_Q \frac{\overline{\langle Q \rangle} Q}{2}\right]$$

$$R, Q \in \{I, \sigma_x, \sigma_y, \sigma_z\}$$

$$\Rightarrow \boxed{\rho = \sum_{R,Q} \frac{\overline{\langle R \otimes Q \rangle} (R \otimes Q)}{4}, \quad R, Q \in \{I, \sigma_x, \sigma_y, \sigma_z\}}$$

There are 16 pairs $R \otimes Q$

→ to reconstruct ρ, 15 measurable quantities $m_i = \overline{\langle R \otimes Q \rangle}$ are required (the 16^{th} constraint is provided by $\mathrm{Tr}\{\rho\} = 1$)

Section 1b.

Defining the Measurement Operators

PRL **102,** 200402 (2009) PHYSICAL REVIEW LETTERS week ending 22 MAY 2009

Two-Qubit State Tomography Using a Joint Dispersive Readout

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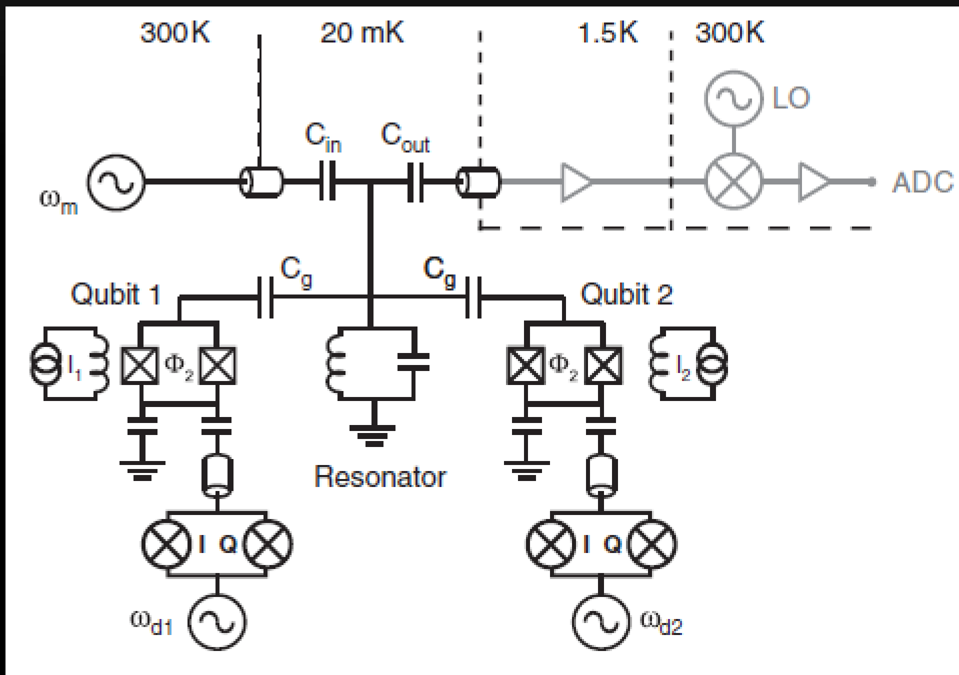


$$H = \hbar(\Delta_{rm} + \chi_1 \hat{\sigma}_{z1} + \chi_2 \hat{\sigma}_{z2})\hat{a}^\dagger \hat{a} + \frac{\hbar}{2} \sum_{j=1,2} (\omega_{aj} + \chi_j)\hat{\sigma}_{zj} + \hbar\epsilon(t)(\hat{a}^\dagger + \hat{a})$$

$$\hat{\chi} \equiv \chi_1 \hat{\sigma}_{z1} + \chi_2 \hat{\sigma}_{z2}$$

The average values of the field quadratures $\langle \hat{I}(t) \rangle = [\hat{\rho}(t)(\hat{a}^\dagger + \hat{a})]$ and $\langle \hat{Q}(t) \rangle = i\,\mathrm{Tr}[\hat{\rho}(t)(\hat{a}^\dagger - \hat{a})]$ are determined from the amplified voltage signal at the resonator output in a homodyne measurement, where $\hat{\rho}(t)$ denotes the state of both qubits and resonator field.

Taking the trace on the resonator space yields $\langle \hat{I}(t)\rangle, \langle \hat{Q}(t)\rangle = \mathrm{Tr}_q[\hat{\rho}_q(0)\hat{M}_{I,Q}(t)]$, where $\hat{M}_{I,Q}(t) = \sum_\sigma \langle \alpha_\sigma(t)|\hat{I}, \hat{Q}|\alpha_\sigma(t)\rangle |\sigma\rangle\langle\sigma|$ and Tr_q denotes the partial trace over the qubits. In the steady state we find

$$\hat{M}_I = -\epsilon \frac{2(\Delta_{rm} + \hat{\chi})}{(\Delta_{rm} + \hat{\chi})^2 + (\kappa/2)^2}, \tag{2}$$

$$\hat{M}_Q = -\epsilon \frac{\kappa}{(\Delta_{rm} + \hat{\chi})^2 + (\kappa/2)^2}, \tag{3}$$

demonstrating that the measurement operators are nonlinear functions of $\hat{\chi}$. Thus, $\hat{M}_{I,Q}$ comprises in general also two-qubit correlation terms proportional to $\hat{\sigma}_{z1}\hat{\sigma}_{z2}$, which allow one to reconstruct the full two-qubit state.

In our experiments the phase of the measurement microwave at frequency $\Delta_{rm} = (\chi_1 + \chi_2)$ is adjusted such that the Q quadrature of the transmitted signal carries most of the signal when both qubits are in the ground state. The corresponding measurement operator can be expressed as

$$\hat{M} = \tfrac{1}{4}(\beta_{00}\hat{\mathrm{id}} + \beta_{10}\hat{\sigma}_{z1} + \beta_{01}\hat{\sigma}_{z2} + \beta_{11}\hat{\sigma}_{z1}\hat{\sigma}_{z2}), \tag{4}$$

with $\beta_{ij} = \alpha_{--} + (-1)^j\alpha_{-+} + (-1)^i\alpha_{+-} + (-1)^{i+j}\alpha_{++}$ and

$$\alpha_{\pm\pm} = -\kappa\{(\kappa/2)^2 + (\Delta_{rm} \pm \chi_1 \pm \chi_2)^2\}^{-1/2} \tag{5}$$

representing the qubit state dependent Q-quadrature amplitudes of the resonator field in the steady-state limit and for an infinite qubit lifetime [Fig. 2(a)].

Thus the measurement operation (along a single quadrature) has the general form:

$$M = \beta_{00}\mathrm{I}\otimes\mathrm{I} + \beta_{10}\mathrm{Z}\otimes\mathrm{I} + \beta_{01}\mathrm{I}\otimes\mathrm{Z} + \beta_{11}\mathrm{Z}\otimes\mathrm{Z}$$

where the physical observable is the homodyne voltage, and the coefficients βij are obtained from calibration (see following slides).

The measurement axes can be changed (i.e. to probe X⊗Y, etc) by pre-rotating the qubits immediately prior to readout.

CALIBRATION…

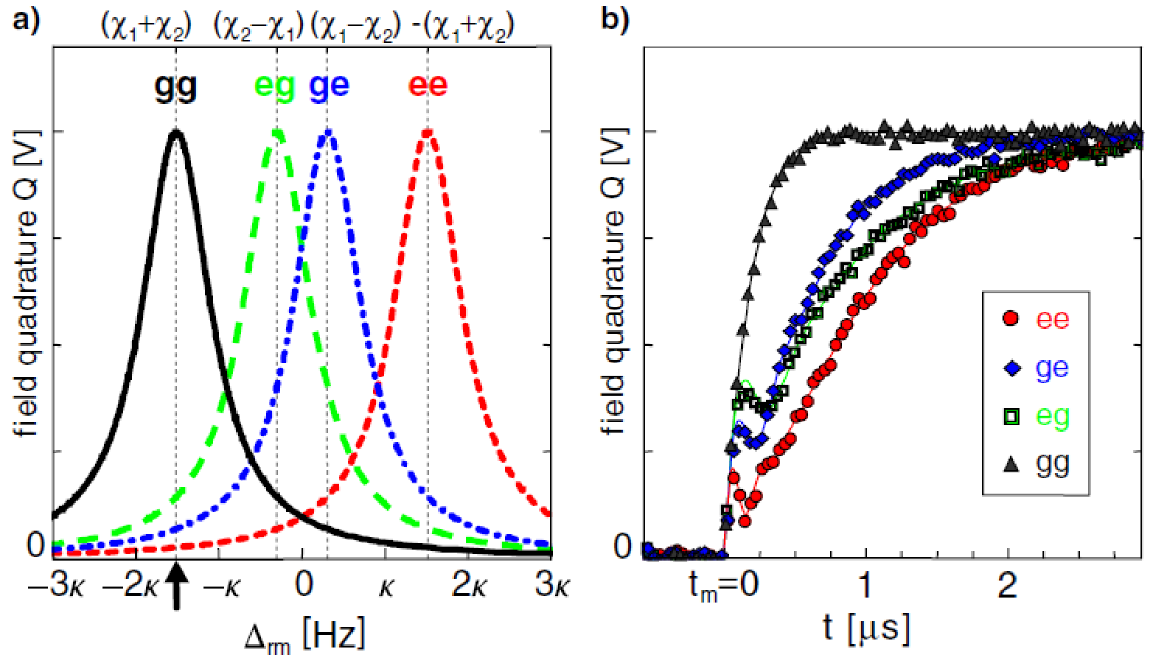
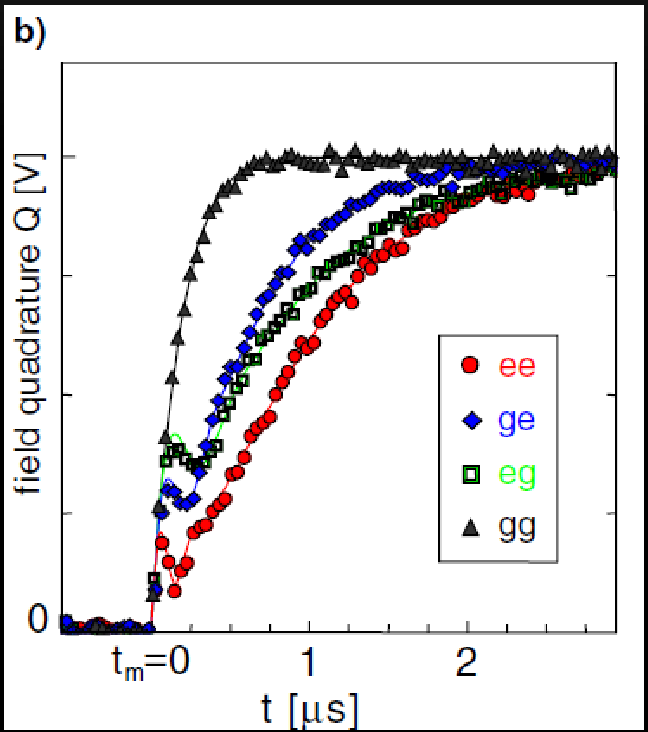


FIG. 2 (color online). (a) Q quadrature of the resonator field for the qubits in states gg, eg, ge, and ee as a function of the detuning Δ_{rm}. Tomography measurements have been performed at $\Delta_{rm} = (\chi_1 + \chi_2)$ indicated by an arrow. (b) Measured (data points) time evolution of the Q quadrature for the indicated initial states compared to numerically calculated responses (solid lines). All parameters have been determined in independent measurements.



consequently limits the readout time to $\sim 1/\gamma_1$. A typical averaged time trace of the resonator response for pulsed measurements is shown in Fig. 2(b), similar to the data presented in Ref. [24]. The qubits are prepared initially in the states $|ee\rangle$, $|eg\rangle$, $|ge\rangle$, and $|gg\rangle$, respectively, using the local gate lines. The time dependence of the measurement signal is determined by the rise time of the resonator and the decay time of the qubits. It is in excellent agreement with calculations [solid lines in Fig. 2(b)] of the dynamics of the dispersive Jaynes-Cummings Hamiltonian [32,34] using the parameter values as stated above. Because of the quantum nondemolition nature of the measurement [27], $\hat{M}$ remains diagonal in the instantaneous qubit eigenbasis during the measurement process, and the integrated signal can be used to define the realistic measurement operator $\hat{M}'$ by replacing the $\alpha_{\pm\pm}$ in Eq. (5) with the signal integrated from the start of the measurement t_m to the final time T, $\alpha'_{\pm\pm} = 1/N \int_{t_m}^{T} [\langle \hat{M}(t)\rangle_{\pm\pm} - \langle \hat{M}(t)\rangle_{--}]dt$ with the ground state response $\langle \hat{M}(t)\rangle_{--}$ subtracted. The normalization constant N is chosen such that $\alpha'_{+-} = 1$ and the measurement time $T - t_m = 2\ \mu$s.

To determine the measurement operator $\hat{M}'$, π pulses are alternately applied to both qubits to yield signals as shown in Fig. 2(b). From these data the coefficients $(\beta'_{00}, \beta'_{01}, \beta'_{10}, \beta'_{11}) = (0.8, -0.3, -0.4, -0.1)$ of $\hat{M}'$ in Eq. (4) are deduced. The nonvanishing β'_{11}, which quantifies the contribution of the $\hat{\sigma}_z \otimes \hat{\sigma}_z$ two-qubit correlation term, allows for a measurement of arbitrary, entangled and separable, quantum states.

From " Quantum Information Processing with Superconducting Qubits" → PhD Thesis by Jerry M. Chow

Table 8.1: The 30 raw measurements.

	Pre-rotation	Measurement operator
M_{01}	$I\otimes I$	$+\beta_{ZI}ZI+\beta_{IZ}IZ+\beta_{ZZ}ZZ$
M_{02}	$R_x^{\pi}\otimes I$	$-\beta_{ZI}ZI+\beta_{IZ}IZ-\beta_{ZZ}ZZ$
M_{03}	$I\otimes R_x^{\pi}$	$+\beta_{ZI}ZI-\beta_{IZ}IZ-\beta_{ZZ}ZZ$
M_{04}	$R_x^{\pi/2}\otimes I$	$+\beta_{ZI}YI+\beta_{IZ}IZ+\beta_{ZZ}YZ$
M_{05}	$R_x^{\pi/2}\otimes R_x^{\pi/2}$	$+\beta_{ZI}YI+\beta_{IZ}IY+\beta_{ZZ}YY$
M_{06}	$R_x^{\pi/2}\otimes R_y^{\pi/2}$	$+\beta_{ZI}YI-\beta_{IZ}IX-\beta_{ZZ}YX$
M_{07}	$R_x^{\pi/2}\otimes R_x^{\pi}$	$+\beta_{ZI}YI-\beta_{IZ}IZ-\beta_{ZZ}YZ$
M_{08}	$R_y^{\pi/2}\otimes I$	$-\beta_{ZI}XI+\beta_{IZ}IZ-\beta_{ZZ}XZ$
M_{09}	$R_y^{\pi/2}\otimes R_x^{\pi/2}$	$-\beta_{ZI}XI+\beta_{IZ}IY-\beta_{ZZ}XY$
M_{10}	$R_y^{\pi/2}\otimes R_y^{\pi/2}$	$-\beta_{ZI}XI-\beta_{IZ}IX+\beta_{ZZ}XX$
M_{11}	$R_y^{\pi/2}\otimes R_x^{\pi}$	$-\beta_{ZI}XI-\beta_{IZ}IZ+\beta_{ZZ}XZ$
M_{12}	$I\otimes R_x^{\pi/2}$	$+\beta_{ZI}ZI+\beta_{IZ}IY+\beta_{ZZ}ZY$
M_{13}	$R_x^{\pi}\otimes R_x^{\pi/2}$	$-\beta_{ZI}ZI+\beta_{IZ}IY-\beta_{ZZ}ZY$
M_{14}	$I\otimes R_y^{\pi/2}$	$+\beta_{ZI}ZI-\beta_{IZ}IX-\beta_{ZZ}ZX$
M_{15}	$R_x^{\pi}\otimes R_y^{\pi/2}$	$-\beta_{ZI}ZI-\beta_{IZ}IX+\beta_{ZZ}ZX$

N_{01}	$I\otimes I$	$+\beta_{ZI}ZI+\beta_{IZ}IZ+\beta_{ZZ}ZZ$
N_{02}	$R_x^{-\pi}\otimes I$	$-\beta_{ZI}ZI+\beta_{IZ}IZ-\beta_{ZZ}ZZ$
N_{03}	$I\otimes R_x^{-\pi}$	$+\beta_{ZI}ZI-\beta_{IZ}IZ-\beta_{ZZ}ZZ$
N_{04}	$R_x^{-\pi/2}\otimes I$	$-\beta_{ZI}YI+\beta_{IZ}IZ-\beta_{ZZ}YZ$
N_{05}	$R_x^{-\pi/2}\otimes R_x^{-\pi/2}$	$-\beta_{ZI}YI-\beta_{IZ}IY+\beta_{ZZ}YY$
N_{06}	$R_x^{-\pi/2}\otimes R_y^{-\pi/2}$	$-\beta_{ZI}YI+\beta_{IZ}IX-\beta_{ZZ}YX$
N_{07}	$R_x^{-\pi/2}\otimes R_x^{-\pi}$	$-\beta_{ZI}YI-\beta_{IZ}IZ+\beta_{ZZ}YZ$
N_{08}	$R_y^{-\pi/2}\otimes I$	$+\beta_{ZI}XI+\beta_{IZ}IZ+\beta_{ZZ}XZ$
N_{09}	$R_y^{-\pi/2}\otimes R_x^{-\pi/2}$	$+\beta_{ZI}XI-\beta_{IZ}IY-\beta_{ZZ}XY$
N_{10}	$R_y^{-\pi/2}\otimes R_y^{-\pi/2}$	$+\beta_{ZI}XI+\beta_{IZ}IX+\beta_{ZZ}XX$
N_{11}	$R_y^{-\pi/2}\otimes R_x^{-\pi}$	$+\beta_{ZI}XI-\beta_{IZ}IZ-\beta_{ZZ}XZ$
N_{12}	$I\otimes R_x^{-\pi/2}$	$+\beta_{ZI}ZI-\beta_{IZ}IY-\beta_{ZZ}ZY$
N_{13}	$R_x^{-\pi}\otimes R_x^{-\pi/2}$	$-\beta_{ZI}ZI-\beta_{IZ}IY+\beta_{ZZ}ZY$
N_{14}	$I\otimes R_y^{-\pi/2}$	$+\beta_{ZI}ZI+\beta_{IZ}IX+\beta_{ZZ}ZX$
N_{15}	$R_x^{-\pi}\otimes R_y^{-\pi/2}$	$-\beta_{ZI}ZI+\beta_{IZ}IX-\beta_{ZZ}ZX$

3 qubits require 63 measurement operators

4 qubits require 255… (goes as 4^N – 1)

Two-Qubit $\underset{\sim}{M}$ matrix

(II excluded)

$$\begin{array}{c} \vec{M} \\ \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \\ M_7 \\ M_8 \\ M_9 \\ M_{10} \\ M_{11} \\ M_{12} \\ M_{13} \\ M_{14} \\ M_{15} \end{bmatrix} \end{array} = \begin{array}{ccc|cccc|cccc|cccc} IX & IY & IZ & XI & XX & XY & XZ & YI & YX & YY & YZ & ZI & ZX & ZY & ZZ \\ \hline & & \beta_2 & & & & & & & & & \beta_1 & & & \beta_3 \\ & & \beta_2 & & & & & & & & & -\beta_1 & & & -\beta_3 \\ & & -\beta_2 & & & & & & & & & \beta_1 & & & -\beta_3 \\ & & \beta_2 & & & & & \beta_1 & & & \beta_3 & & & & \\ & \beta_2 & & & & & & \beta_1 & & \beta_3 & & & & & \\ -\beta_2 & & & & & & & \beta_1 & -\beta_3 & & & & & & \\ & & -\beta_2 & & & & & \beta_1 & & & -\beta_3 & & & & \\ & & \beta_2 & -\beta_1 & & & -\beta_3 & & & & & & & & \\ & \beta_2 & & -\beta_1 & & -\beta_3 & & & & & & & & & \\ -\beta_2 & & & -\beta_1 & \beta_3 & & & & & & & & & & \\ & & -\beta_2 & -\beta_1 & & & \beta_3 & & & & & & & & \\ & \beta_2 & & & & & & & & & & \beta_1 & & \beta_3 & \\ & \beta_2 & & & & & & & & & & -\beta_1 & & -\beta_3 & \\ -\beta_2 & & & & & & & & & & & \beta_1 & -\beta_3 & & \\ -\beta_2 & & & & & & & & & & & -\beta_1 & \beta_3 & & \end{array} \begin{array}{c} \vec{N} \\ \begin{bmatrix} IX \\ IY \\ IZ \\ XI \\ XX \\ XY \\ XZ \\ YI \\ YX \\ YY \\ YZ \\ ZI \\ ZX \\ ZY \\ ZZ \end{bmatrix} \end{array}$$

All empty entries are ZERO

$M_1 - M_{15}$ as defined on page 207 of Jerry Chow's Thesis

Section 1c.

In Practice…

Now, using the $\overline{\langle R \otimes Q \rangle}$ we can obtain ρ directly using

$$\rho = \sum_{rp} \frac{\overline{\langle R \otimes Q \rangle} (R \otimes Q)}{4}$$

however, due to measurement imperfections and noise, this will not necessarily produce a Hermitian, positive-definite state! ($\therefore$ is not necessarily "physical").

Probabilities must be real and non-negative!

PHYSICAL REVIEW A, VOLUME 64, 052312

Measurement of qubits

Daniel F. V. James,[1,*] Paul G. Kwiat,[2,3] William J. Munro,[4,5] and Andrew G. White[2,4]

A ubiquitous remedy: Maximum Likelihood Approximation (MLE)

The definition ensuring the desired properties comes from the "Cholesky Decomposition"

$$\star \quad \rho_p = \frac{\hat{T}\hat{T}^\dagger}{\mathrm{Tr}\{\hat{T}\hat{T}^\dagger\}}$$

(subscript "p" denotes "physical")

where (for a 4x4 density matrix and 15 independent real parameters)

$$\hat{T}(t) = \begin{bmatrix} t_1 & 0 & 0 & 0 \\ t_5 + it_6 & t_2 & 0 & 0 \\ t_{11} + it_{12} & t_7 + it_8 & t_3 & 0 \\ t_{15} + it_{16} & t_{13} + it_{14} & t_9 + it_{10} & t_4 \end{bmatrix}$$

The t_i are parameters to be optimized during MLE.

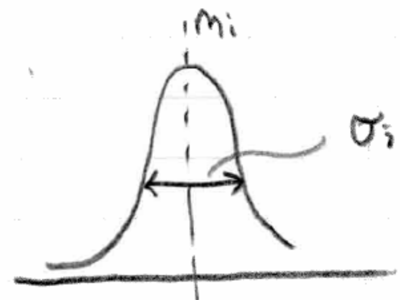
Any given ρ_p will have expectation values for each of the measurement operators given by $\overline{\langle M_i^p \rangle} = \mathrm{Tr}\{M_i \rho_p\}$. These can be probabilistically contrasted with the set of experimentally obtained measurement operators $\langle M_i^E \rangle = m_i$ using Gaussian statistics.

i.e. for any given expectation value (ensemble average)

$$P_i(m_i) = \frac{1}{\text{Normalization}} \, e^{-\frac{(m_i - \mathrm{Tr}\{M_i \rho_p\})^2}{2\sigma_i^2}}$$

probability

i.e.



σ_i taken to be approx. $\sqrt{\mathrm{Tr}\{M_i \rho_p\}}$ (most commonly)

For <u>all</u> measurement values, the combined probability is the product of the individual probabilities

$$P = \frac{1}{Norm} \prod_i \exp\left\{ \frac{-(m_i - Tr\{M_i \rho_p\})^2}{2Tr\{M_i \rho_p\}} \right\}$$

This is the function we seek to maximize for some choice of t_i. It is, however, easier to maximize its logarithm:

$$\ln(P) = \ln\left\{ \frac{1}{Norm} \left(\exp\left(\frac{-(m_1 - Tr(M_1\rho_p))^2}{2Tr(M_1\rho_p)}\right)\right)\left(\exp\left(\frac{-(m_2 - Tr(M_2\rho_p))^2}{2Tr(M_2\rho_p)}\right)\right)\cdots \right\}$$

$$= \ln\left\{\frac{1}{Norm}\right\} - \frac{(m_1 - Tr(M_1\rho_p))^2}{2Tr(M_1\rho_p)} - \frac{(m_2 - Tr(M_2\rho_p))^2}{2Tr(M_2\rho_p)} - \cdots$$

$$= \ln\left(\frac{1}{Norm}\right) - \underbrace{\sum_i \left[\frac{(m_i - Tr\{M_i\rho_p\})^2}{2Tr\{M_i \rho_p\}} \right]}_{}$$

ln(P) maximized by minimizing this

∴ Our MLE proceeds by <u>minimizing</u> the function

$$\boxed{\mathcal{L}(t_1, t_2, \cdots, t_{16}) = \sum_{i=1}^{15} \frac{(m_i - Tr\{M_i \rho_p\})^2}{2 Tr\{M_i \rho_p\}}}$$

Which is subject to the additional constraint that $Tr\{\rho_p\} = 1$.

To provide the algorithm with starting values for the t_i …

■ **APPROXIMATE CHOLESKY DECOMPOSITION**

```
(*NULLIFY NEGATIVE EIGENVALUES TO APPROXIMATE MATRIX AS POSITIVE SEMI-DEFINITE AND DETERMINE STARTING VALUES FOR T*)
(*see QPT and Linblad estimation of SSQb paper*)

(*DIAGONALIZE THE MATRIX*)
Evects = Eigenvectors[rhoE];
DIAGrhoE = Inverse[Transpose[Evects]].rhoE.Transpose[Evects] // Chop;
MatrixForm[DIAGrhoE]

(*REMOVE ALL NON-POSITIVE EIGENVALUES*)
For[i = 1, i < 5, i++, If[DIAGrhoE[[i, i]] <= 0, DIAGrhoE[[i, i]] = epsilon]]
MatrixForm[DIAGrhoE - epsilon * IdentityMatrix[4]] // Chop;

(*INVERSE THE TRANSFORMATION*)
APRXrhoE = Transpose[Evects].DIAGrhoE.Inverse[Transpose[Evects]] // Chop;
APRXrhoE = SetPrecision[APRXrhoE, 14];(*Roundoff error causes the Cholesky Decomposition to think that this matrix is n
MatrixForm[Round[APRXrhoE, 0.00001]];

(*PERFORM THE CHOLESKY DECOMPOSITION TO OBTAIN STARTING VALUES*)
Tstart = CholeskyDecomposition[APRXrhoE] // ConjugateTranspose;
```

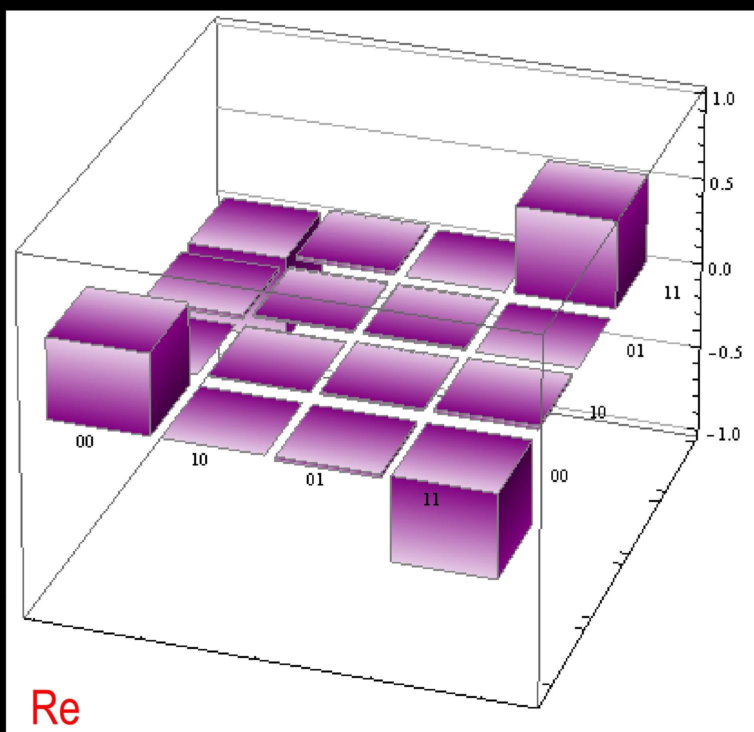
USER INPUTS

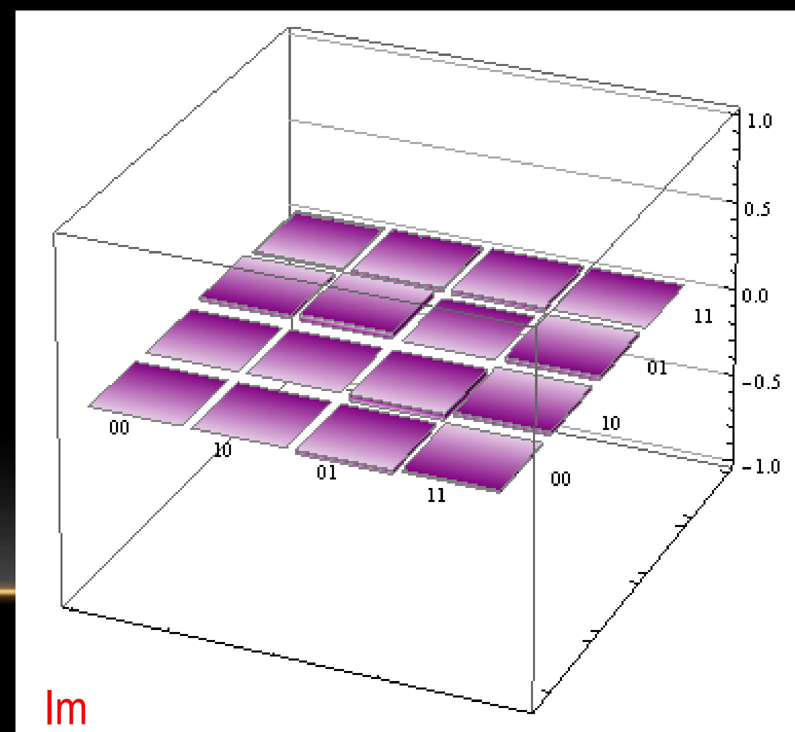
```
(*AVERAGE MEASURED VALUES*)
Minput = {2.05, 0.02, 0.05, 0.95, 1.95, 0.95, 1.05, 0.95, 1.05, 0.05, 1.05, 0.95, 0.95, 1.05, 0.95};


(*CALIBRATION CONSTANTS*)
BII = 1;
BIZ = 1;
BZI = 1;
BZZ = 1;
```

Corresponds to $|\varphi\rangle = \frac{1}{\sqrt{2}}[|00\rangle - |11\rangle]$

(plus some experimental error)

$$\begin{pmatrix} 0.47 & 2.77556\times10^{-17} + 2.77556\times10^{-17}\,i & -0.02125 - 0.02125\,i & -0.4875 + 0.0125\,i \\ 2.77556\times10^{-17} - 2.77556\times10^{-17}\,i & 0.005 & 0.0125 - 0.0375\,i & 0.02125 + 0.02125\,i \\ -0.02125 + 0.02125\,i & 0.0125 + 0.0375\,i & 0.0125 & -2.77556\times10^{-17} + 0.025\,i \\ -0.4875 - 0.0125\,i & 0.02125 - 0.02125\,i & -2.77556\times10^{-17} - 0.025\,i & 0.5125 \end{pmatrix}$$

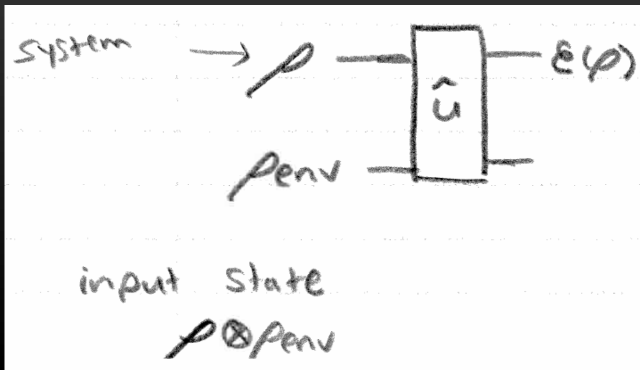




Part 2: Quantum Process Tomography (QPT)

Section 2a.

Operator-Sum Representation



$$\boxed{\mathcal{E}(\rho) = \mathrm{Tr}_{env}\left\{\hat{U}(\rho \otimes \rho_{env})\hat{U}^\dagger\right\}}$$

Assumes no initial correlation between system and environment.

II) Operator-Sum Representation:

Following from I), let $\rho_{env} = |e_0\rangle\langle e_0|$ be the initial state of the environment (either a mixed or pure state*) and let $|e_k\rangle$ be an orthonormal basis for the finite-dimensional state space of the environment. ρ is the principal system of interest.

$$\Rightarrow \mathcal{E}(\rho) = \mathrm{Tr}_{env}\left\{\hat{U}(\rho \otimes |e_0\rangle\langle e_0|)\hat{U}^\dagger\right\}$$

$$= \sum_k \langle e_k|\hat{U}(\rho \otimes |e_0\rangle\langle e_0|)\hat{U}^\dagger|e_k\rangle$$

$$\boxed{\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger}$$

where $E_k = \langle e_k|\hat{U}|e_0\rangle \equiv$ operation elements for $\mathcal{E}$

See page 361 of Nielsen and Chuang

A proof (based on Gernot Pfanner, pg 33 of Thesis “A study on photon-entanglement from the biexciton cascade in a quantum dot”)

We start with an open quantum system S, which is initially disentangled from its environment, i.e.

$$\rho_{SE}(t_0) = \rho_S \otimes \rho_E$$

The state of the environment at $t = t_0$ should be $\rho_E = |e_0\rangle \langle e_0|$.
Performing a unitary operation $\hat{U}$ on the system-environment-complex yields

$$\mathcal{E}(\rho) = Tr_E \left(U \left\{ \rho_S \otimes \rho_E \right\} U^\dagger \right) \tag{1}$$

The operation $\hat{U}$ may be written as

$$\hat{U} = \sum_i \alpha_i \hat{S}_i \otimes \hat{T}_i$$

where $\hat{S}_i$ and $\hat{T}_i$ are linear operators on $\mathcal{H}_S$ and $\mathcal{H}_E$ respectively.
By inserting this into equation (1), we obtain

$$\begin{aligned}
\mathcal{E}(\rho) &= \sum_k \langle e_k| \sum_i \left(\alpha_i \hat{S}_i \otimes \hat{T}_i \right) (\rho_S \otimes \rho_E) \sum_j \left(\alpha_j \hat{S}_j \otimes \hat{T}_j \right)^\dagger |e_k\rangle \\
&= \sum_{ijk} \alpha_i \alpha_j^* \langle e_k| \left(\hat{S}_i \rho_S \hat{S}_j^\dagger \otimes \hat{T}_i \rho_E \hat{T}_j^\dagger \right) |e_k\rangle \\
&= \sum_{ijk} \alpha_i \alpha_j^* \hat{S}_i \rho_S \hat{S}_j^\dagger \langle e_k| \hat{T}_i |e_0\rangle \langle e_0| \hat{T}_j^\dagger |e_k\rangle \\
&= \sum_k \left(\sum_i \alpha_i \hat{S}_i \langle e_k| \hat{T}_i |e_0\rangle \right) \rho_S \left(\sum_j \alpha_j \hat{S}_j \langle e_k| \hat{T}_j |e_0\rangle \right)^\dagger \\
&= \sum_k \left(\langle e_k| \sum_i \alpha_i \hat{S}_i \otimes \hat{T}_i |e_0\rangle \right) \rho_S \left(\langle e_k| \sum_j \alpha_j \hat{S}_j \otimes \hat{T}_j |e_0\rangle \right)^\dagger \\
&= \sum_k \langle e_k| \hat{U} |e_0\rangle \rho_S \langle e_k| \hat{U} |e_0\rangle^\dagger
\end{aligned}$$

and we therefore arrive at

$$\mathcal{E}(\rho) = \sum_k \hat{E}_k \rho_S \hat{E}_k^\dagger \qquad \hat{E}_k = \langle e_k| \hat{U} |e_0\rangle$$

$$\mathcal{E}(\rho) = \sum_k \hat{E}_k \rho_S \hat{E}_k^\dagger \qquad \hat{E}_k = \langle e_k | \hat{U} | e_0 \rangle$$

AXIOMATIC APPROACH TO QUANTUM OPERATORS:

We define a quantum operation $\mathcal{E}$ as a map from the set of density operators of the input space Q_1 to the set of density operators for the output space Q_2.

$$\mathcal{E}(\rho) = \sum_k \hat{E}_k \rho_S \hat{E}_k^\dagger \qquad \hat{E}_k = \langle e_k | \hat{U} | e_0 \rangle$$

Iff $\mathcal{E}$ satisfies the following axioms, it has an operator-sum representation:

(for simplicity take $Q_1 = Q_2 = Q$)

A1: $\mathrm{Tr}\{\mathcal{E}(\rho)\}$ is the probability that the process represented by $\mathcal{E}$ occurs, when ρ is the initial state. Thus $0 \leq \mathrm{Tr}\{\mathcal{E}(\rho)\} \leq 1$ for any ρ.

A2: $\mathcal{E}$ is a "convex-linear" map on the set of density matrices. That is, for probabilities $\{p_i\}$

$$\mathcal{E}\left(\sum_i p_i \rho_i\right) = \sum_i p_i \mathcal{E}(\rho_i)$$

A3: $\mathcal{E}$ is a completely positive map
$\rightarrow \mathcal{E}(A) > 0$ for any operator $\hat{A}$

(the correctly normalized quantum state is therefore $\mathcal{E}(\rho) / \mathrm{Tr}[\mathcal{E}(\rho)]$

Other Comments

$\mathcal{E}(\rho)$, $F(\rho)$

The operation elements E_K, F_K appearing in an operator-sum representation are not unique.

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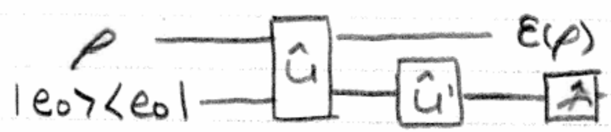
- different physical processes can give rise to the same system dynamics



↳ subsequent transform on environment doesn't affect the principal state
→ $\mathcal{E}(\rho)$ unchanged.

- as a result of this, all quantum operations $\mathcal{E}$ on a system w/ dimension "d" (i.e. its Hilbert Space) can be generated by an operator-sum representation containing at most d^2 elements:

$$\mathcal{E}(\rho) = \sum_{K=1}^{M} E_K \rho E_K^{\dagger} \qquad (1 \leq M \leq d^2)$$

Section $\frac{1}{\sqrt{2}}\left[|2b.\rangle + |\overline{2b.}\rangle\right]$

QPT – The "Black-Box" Recipe

Prescription for experimental determination of the dynamics of a quantum black box

Isaac L. Chuang [1,2] *and* M. A. Nielsen [1,3]

The original recipe (Journal of Modern Optics, 1997)

Quantum Process Tomography

From operator-sum theory, the evolution of a density matrix may be written as the mapping:

$$\mathcal{E}(\rho) = \sum_i \hat{A}_i \rho \hat{A}_i^\dagger$$

If we knew the exact process undergone by ρ, then in principle we could determine the $\hat{A}_i$ from theory. Experimentally, however, we encounter scenarios where all we know are the input states and output states of a quantum 'black box' which we wish to characterize/create a mapping for.

- the $\hat{A}_i$ have some "representational freedom" and are not necessarily unique
- it is thus important to construct $\hat{A}_i$ from a fixed set of operators $\tilde{A}_i$

↳ (i.e. a Pauli representation)

$$\Rightarrow \boxed{A_i = \sum_m a_{im} \tilde{A}_m}$$

Thus we have

$$\boxed{\mathcal{E}(\rho) = \sum_{m,n} \chi_{mn} \tilde{A}_m \rho \tilde{A}_n^\dagger}$$

"The matrix χ completely and uniquely describes the process $\mathcal{E}$ and can be reconstructed from experimental tomographic measurements."

[The trace-preserving constraint is

$$\sum_i A_i^\dagger A_i = I \rightarrow \sum_{m,n} \chi_{mn} \tilde{A}_n^\dagger \tilde{A}_m = I$$
]

We must choose a complete and linearly independent basis of "input" states ρ_i (we require d^2 of them, where d is the dimension of the density matrix)

Each output state $\mathcal{E}(\rho_i)$ can be expressed as a linear combination of our input states (since our basis was complete):

$$\mathcal{E}(\rho_i) = \sum_k \lambda_{ik} \rho_k$$

- we input ρ_i, we determine $\mathcal{E}(\rho_i)$ from quantum state tomography (QST) and then we use the above relation to determine the matrix λ_{ik}.

If we write $\tilde{A}_m \rho_j \tilde{A}_n^\dagger = \sum_k \beta_{jk}^{mn} \rho_k$
(where β_{jk} can be determined from our choice of $\tilde{A}_m$ and ρ_i)
then:

$$\mathcal{E}(\rho) = \sum_{m,n} \chi_{mn} \tilde{A}_m \rho \tilde{A}_n^\dagger, \qquad \tilde{A}_m \rho_j \tilde{A}_n^\dagger = \sum_k \beta_{jk}^{mn} \rho_k$$

$$\mathcal{E}(\rho_j) = \sum_k \lambda_{jk} \rho_k$$

$$\sum_k \sum_{mn} \chi_{mn} \beta_{jk}^{mn} \rho_k = \sum_k \lambda_{jk} \rho_k$$

$\Downarrow$ since ρ_k is "independent"

$$\boxed{\sum_{mn} \beta_{jk}^{mn} \chi_{mn} = \lambda_{jk}}$$

Thus, x_{mn} is obtained from the matrix "λ" by operating on λ with the <u>pseudoinverse</u> of β.

A pseudo (or generalized) inverse "K" is defined by:

$$\beta_{jk}^{mn} = \sum_{st,xy} \beta_{jk}^{st} K_{st}^{xy} \beta_{xy}^{mn}$$

From this we obtain $x_{mn} = \sum_{jk} K_{jk}^{mn} \lambda_{jk}$

Finally, the "operator-sum" representation for $\mathcal{E}$ can be obtained from

$$\hat{A}_i = \sqrt{d_i} \sum_j U_{ij} \tilde{A}_j$$

where $\hat{U}^{+}$ is a unitary matrix which diagonalizes $\mathcal{X}$ such that

$$\mathcal{X}_{mn} = \sum_{xy} U_{mx} \underbrace{d_x \, \delta_{xy}}_{D_{xy}} U_{ny}^{*}$$

Section 2c.

Implementation of Single-Qubit QPT

① Prepare a complete basis of input states

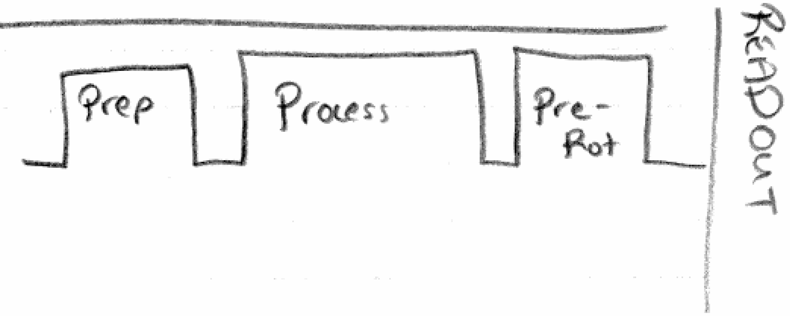
i.e. $\rho_j = |\psi_j\rangle\langle\psi_j|$, $|\psi_j\rangle = \{|0\rangle, |1\rangle, \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle], \frac{1}{\sqrt{2}}[|0\rangle + i|1\rangle]\}$

for a single qubit

② Apply the unknown process $\mathcal{E}$ to each member of ρ_j

③ Reconstruct the output states $\mathcal{E}(\rho_j)$ via QST.

Script: 3 control segments:



Prep:

$|0\rangle$

$|1\rangle$

$|+\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$

$|i\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + i|1\rangle\right]$

(for two qubits, $\rho_j = R \otimes Q$: $R, Q \in \{|0\rangle, |1\rangle, |+\rangle, |-\rangle\}$)

$\longrightarrow$ A complete operator basis is

$$\tilde{A}_m \in \{I, \sigma_x, -i\sigma_y, \sigma_z\}$$

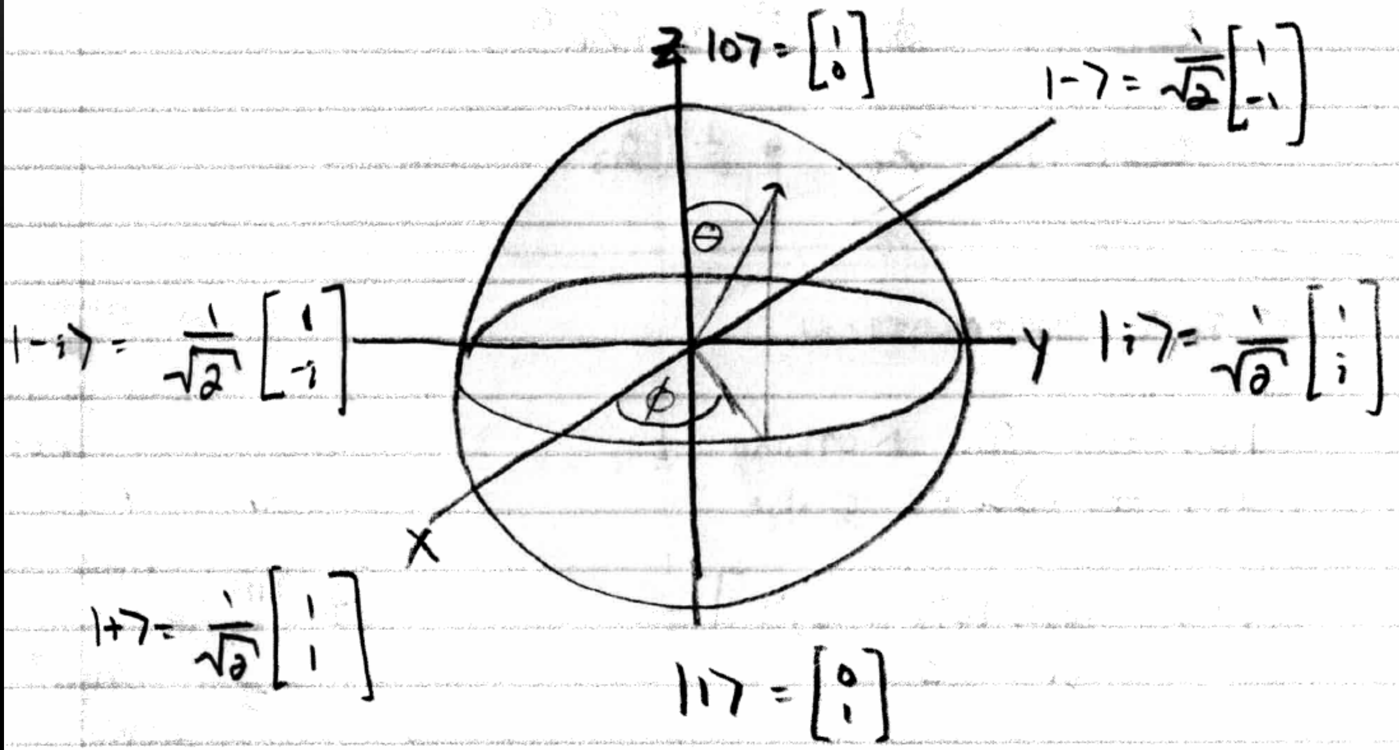
for each qubit.

output of measurement scripts will yield

$$\{ \underbrace{\{m_1, m_2, m_3\}}_{|0\rangle}, \underbrace{\{m_1, m_2, m_3\}}_{|1\rangle}, \underbrace{\{m_1, m_2, m_3\}}_{|+\rangle}, \underbrace{\{m_1, m_2, m_3\}}_{|i\rangle} \}$$

Corresponding Input

From this, QST reconstructs $\mathcal{E}(\rho_i)$



so an $\sigma_x \equiv X$ gate circles CW around the Y-axis
an $\sigma_y \equiv Y$ gate circles CW around the X-axis
an $\sigma_z \equiv Z$ gate circles CW around the Z-axis

$$\rho_{|0\rangle} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \rho_{|1\rangle} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho_{|+\rangle} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \rho_{|-\rangle} = \frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix},$$

$$\begin{bmatrix} \mathcal{E}(\rho_{|0\rangle}) \\ \mathcal{E}(\rho_{|1\rangle}) \\ \mathcal{E}(\rho_{|+\rangle}) \\ \mathcal{E}(\rho_{|-\rangle}) \end{bmatrix} = \begin{bmatrix} & & \\ & \lambda & \\ & & \end{bmatrix} \begin{bmatrix} \rho_{|0\rangle} \\ \rho_{|1\rangle} \\ \rho_{|+\rangle} \\ \rho_{|-\rangle} \end{bmatrix} \qquad \left(\text{i.e. } \mathcal{E}(\rho_j) = \sum_k \lambda_{jk}\rho_k\right)$$

i indicates the ROW

$$\rightarrow \mathcal{E}(\rho_i) = \lambda_1^{(i)}\rho_{|0\rangle} + \lambda_2^{(i)}\rho_{|1\rangle} + \lambda_3^{(i)}\rho_{|+\rangle} + \lambda_4^{(i)}\rho_{|-\rangle}$$

$$\mathcal{E}_{11}(\rho_i) = \lambda_1^{(i)} + \frac{1}{2}\left(\lambda_3^{(i)} + \lambda_4^{(i)}\right)$$

$$\mathcal{E}_{12}(\rho_i) = \frac{1}{2}\left(\lambda_3^{(i)} - i\lambda_4^{(i)}\right)$$

$$\mathcal{E}_{21}(\rho_i) = \frac{1}{2}\left(\lambda_3^{(i)} + i\lambda_4^{(i)}\right)$$

$$\mathcal{E}_{22}(\rho_i) = \lambda_2^{(i)} + \frac{1}{2}\left(\lambda_3^{(i)} + \lambda_4^{(i)}\right)$$

After obtaining the matrix λ, we must determine the matrix β which is specified by our chosen set of input states $\{\rho_j\}$ and operators $\{\tilde{A}_m\}$ with the relation:

$$\tilde{A}_m \rho_j \tilde{A}_n^{\dagger} = \sum_K \beta_{jK}^{mn} \rho_K$$

β is effectively a $4\times4\times4\times4$ array: (or a 4×4 array of 4×4 arrays)

→ for every $\beta(m,n)$ we have a 4×4 array

$$\beta^{mn} = \begin{array}{c c} & \xleftrightarrow{\quad K \quad} \\ j\big\downarrow & \begin{bmatrix} \beta_{11}, \beta_{12}, \beta_{13}, \beta_{14} \\ \ddots \\ \beta_{44} \end{bmatrix} \end{array}$$

For each (m,n) combination of our chosen operators, we are essentially finding the equivalent of a ‘lambda’ matrix that specifies how possible evolutions of the basis states are constructed (from the basis states)

After obtaining β_{jk}^{mn} (which is specific to our choice of ρ_j and $\tilde{A}_m$, we analytically determine the "generalized inverse" (or pseudoinverse)

since $\sum_{mn} \beta_{jk}^{mn} x_{mn} = \lambda_{jk}$,

$$\boxed{x_{mn} = \sum_{jk} K_{jk}^{mn} \lambda_{jk}}$$

where K is the pseudoinverse of β.

C&N note that the pseudoinverse is defined by

$$\beta_{jk}^{mn} = \sum_{st,xy} \beta_{jk}^{st} K_{st}^{xy} \beta_{xy}^{mn}$$

Obtaining a pseudoinverse directly from a matrix of matrices is not an easy task, hence I employ a trick…

Using the idea of a "composite index", we can map a 2D array $M[[i, j]]$ to a 1D vector $V[[N*(i-1)+j]]$ where N is the dimension of matrix M.

Hence, in the case of a single qubit, the 4×4 matrix λ_{jk} is mapped to a 16-element vector that is indexed by $[[4(j-1)+k]]$.

Likewise, the $4\times4\times4\times4$ entity β_{jk}^{mn} is mapped to the 16×16 matrix $B_{(4(j-1)+k),(4(m-1)+n)}$

Likewise, the $4\times4\times4\times4$ entity β_{jk}^{mn} is mapped to the 16×16 matrix $\beta_{(4(j-1)+k),(4(m-1)+n)}$

The pseudoinverse is now easily obtained with a computational software package (i.e. in Mathematica:
inv = PseudoInverse[beta]

With the pseudoinverse K (still in 16×16 form) we may obtain χ (in 16×1 form) by performing the matrix multiplication:

$$\underset{[16\times1]}{\chi} = \underset{[16\times16]}{K} \, . \, \underset{[16\times1]}{\lambda}$$

We may then convert χ back to matrix form ie. with a loop "For[i=1, i<5, i++, For[j=1, j<5, j++, Chi[[i,j]] = Chi1D[[4*(i-1)+j]]]]" in Mathematica.

Having obtained χ, we can determine the Kraus operators $\hat{A}_i$ (i.e. $\mathcal{E}(\rho) = \sum_i \hat{A}_i \rho \hat{A}_i^{\dagger}$) via

$$\hat{A}_i = \sqrt{d_i} \sum_j U_{ji} \tilde{A}_j$$

where d_i are the eigenvalues of χ as obtained through the unitary transformation $D = U^{\dagger} \chi U$

The latter only works if χ is positive semi-definite. To ensure this, it is common to use Maximum Likelihood Estimation (MLE) as was the case in QST.

$$\tilde{\mathcal{X}} = TT^{\dagger}$$

$$\tilde{\mathcal{X}} = \begin{bmatrix} t_1 & 0 & 0 & 0 \\ t_5 + it_6 & t_2 & 0 & 0 \\ t_{11} + it_{12} & t_7 + it_8 & t_3 & 0 \\ t_{15} + it_{16} & t_{13} + it_{14} & t_9 + it_{10} & t_4 \end{bmatrix}$$

$$\mathcal{L} = \sum_{a,b} \left[m_{ab} - \mathrm{Tr}\left\{ \hat{M}_b \left(\sum_{m,n} \tilde{\mathcal{X}}_{mn} \tilde{A}_m \rho_a \tilde{A}_n^{\dagger} \right) \right\} \right]^2$$

$$+ \lambda \sum_{k} \left[\mathrm{Tr}\left\{ \sum_{m,n} \tilde{\mathcal{X}}_{mn} \tilde{A}_m (\tilde{A}_k) \tilde{A}_n^{\dagger} \right\} - \mathrm{Tr}\left\{ \tilde{A}_k \right\} \right]^2$$

where we recognize:

$$\sum_{m,n} \tilde{\mathcal{X}}_{mn} \tilde{A}_m \rho_a \tilde{A}_n^{\dagger} = \tilde{\mathcal{E}}(\rho_a)$$

$$\sum_{m,n} \tilde{\mathcal{X}}_{mn} \tilde{A}_m (\tilde{A}_k) \tilde{A}_n^{\dagger} = \tilde{\mathcal{E}}(\tilde{A}_k)$$

(operation elements important for single qubit operations)

Depolarizing Channel	$\sqrt{1-\frac{3p}{4}}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, $\sqrt{\frac{p}{4}}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$, $\sqrt{\frac{p}{4}}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$, $\sqrt{\frac{p}{4}}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$
Amplitude Damping	$\begin{bmatrix}1 & 0\\0 & \sqrt{1-\gamma}\end{bmatrix}$, $\begin{bmatrix}0 & \sqrt{\gamma}\\0 & 0\end{bmatrix}$
Phase Damping	$\begin{bmatrix}1 & 0\\0 & \sqrt{1-\gamma}\end{bmatrix}$, $\begin{bmatrix}0 & 0\\0 & \sqrt{\gamma}\end{bmatrix}$
Phase Flip	$\sqrt{p}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, $\sqrt{1-p}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$
Bit Flip	$\sqrt{p}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, $\sqrt{1-p}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$
Bit-Phase Flip	$\sqrt{p}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, $\sqrt{1-p}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$

Page 397 from N & C; p is a probability, gamma is a numerical parameter; theoretical values can often be found from the Master equation

Bloch Sphere Visualization…

For any arbitrary trace-preserving quantum operation, we have an "affine map" mapping the Bloch Sphere onto itself:

$$\vec{r} \xrightarrow{\mathcal{E}} \vec{r}\,' = M\vec{r} + \vec{c}$$

Where M is a 3×3 real matrix corresponding to a rotation/scaling and $\vec{c}$ is a constant vector corresponding to a translation. From N & C:

We can visualize the map's effect directly (M. Howard et al, QPT and Linblad estimation...) by parametrizing the qubit state as a 4-vector:

$$\rho = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma}) \longleftrightarrow \frac{1}{2}\begin{bmatrix} 1 \\ r_x \\ r_y \\ r_z \end{bmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ \vec{r} \end{pmatrix}$$

In this basis, any trace-preserving evolution takes the form:

$$\mathcal{E} = \begin{pmatrix} 1 & 0 \\ \vec{c} & M \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ c_x & M_{xx} & M_{xy} & M_{xz} \\ c_y & M_{yx} & M_{yy} & M_{yz} \\ c_z & M_{zx} & M_{zy} & M_{zz} \end{bmatrix}$$

Hence $\mathcal{E}(\vec{r}) = \begin{pmatrix} 1 & 0 \\ \vec{c} & M \end{pmatrix}\begin{pmatrix} 1 \\ \vec{r} \end{pmatrix} = \begin{pmatrix} 1 \\ M\vec{r} + \vec{c} \end{pmatrix} = \begin{pmatrix} 1 \\ \vec{r}' \end{pmatrix} = \vec{r}'$

Suppose the Kraus operators $\hat{A}_i$ generating the map are written in the form

$$\hat{A}_i = \alpha_i I + \sum_{k=1}^{3} a_{ik} \sigma_k$$

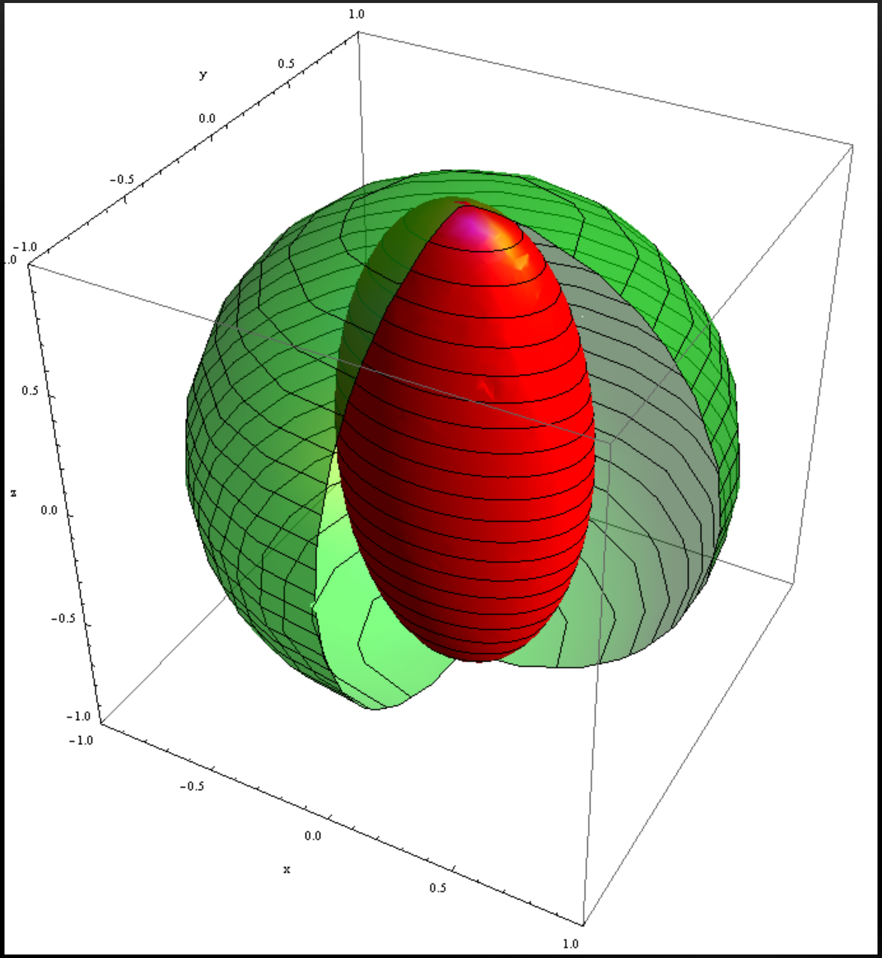
Then:

$$\boxed{M_{jk} = \sum_{\ell} \Big[a_{\ell j} a_{\ell k}^* + a_{\ell j}^* a_{\ell k} + \Big(|\alpha_\ell|^2 - \sum_p a_{\ell p} a_{\ell p}^* \Big) \delta_{jk} + i \sum_p \epsilon_{jkp} \left(\alpha_\ell a_{\ell p}^* - \alpha_\ell^* a_{\ell p} \right) \Big]}$$

and

$$\boxed{c_k = 2i \sum_{\ell} \sum_{jp} \epsilon_{jpk} a_{\ell j} a^*_{\ell p}}$$

It can be shown….



i.e. a dephasing process